ADVANCED PARTIAL DIFFERENTIAL EQUATIONS: HOMEWORK 4

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1. Chapter 5, Problem 2

For convenience set $t := \frac{\gamma-\beta}{1-\beta}$. Then, $\beta = \frac{\gamma-t}{1-t}$ and we see:

$$\begin{aligned}
\frac{|u(x)-u(y)|}{|x-y|^\gamma} &= \frac{|u(x)-u(y)|}{|x-y|^{\gamma-t}|x-y|^t} \\
&= \Big(\frac{|u(x)-u(y)|}{|x-y|^{\frac{\gamma-t}{1-t}}}\Big)^{1-t}\Big(\frac{|u(x)-u(y)|}{|x-y|}\Big)^t \\
&= \Big(\frac{|u(x)-u(y)|}{|x-y|^\beta}\Big)^{1-t}\Big(\frac{|u(x)-u(y)|}{|x-y|}\Big)^t
\end{aligned} \tag{1.1}$$

And taking the supremum over the above:

$$[u]_{C^{0,\gamma}(U)} \leqslant [u]^{1-t}_{C^{0,\beta}(U)}[u]^t_{C^{0,1}(U)}$$

Using this:

$$\begin{aligned}
||u||_{C^{0,\gamma}(U)} &\leqslant ||u||_{C(U)} + [u]^{1-t}_{C^{0,\beta}(U)}[u]^t_{C^{0,1}(U)} \\
&\leqslant \big(||u||^{1-t}_{C(U)} + [u]^{1-t}_{C^{0,\beta}(U)}\big)\big(||u||^t_{C(U)} + [u]^t_{C^{0,1}(U)}\big) \\
&\leqslant \big(||u||_{C(U)} + [u]_{C^{0,\beta}(U)}\big)^{1-t}\big(||u||_{C(U)} + [u]_{C^{0,1}(U)}\big)^t \\
&= ||u||^{\frac{1-\gamma}{1-\beta}}_{C^{0,\beta}(U)}||u||^{\frac{\gamma-\beta}{1-\beta}}_{C^{0,1}(U)}
\end{aligned} \tag{1.2}$$

Where the second to last inequality is standard and is applicable since the sum of the exponents is 1, and the final step uses concavity

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since $t < 1$. Then this is precisely the inequality that we are looking for.

2. Chapter 5, Problem 3

3. Chapter 5, Problem 4

(a). Let v denote the weak derivative of u. Then, $\int_0^x v(t)dt$ is an absolutely continuous function (this is a standard result in real analysis), and for any test function $\phi \in C_0^\infty(U)$:

(3.1)

$$\begin{aligned}\int_0^1 \Big(\int_0^x v(t)dt - u(x) \Big) \phi'(x)dx &= \int_0^1 \int_0^x v(t)\phi'(x)dtdx - \int_0^1 u(x)\phi'(x)dx \\ &= \int_0^1 \int_t^1 \phi'(x)dxv(t)dt - \int_0^1 u(x)\phi'(x)dx \\ &= \int_0^1 \phi(t)v(t)dt - \int_0^1 u(x)\phi'(x)dx \\ &= \int_0^1 \phi(x)v(x)dx - \int_0^1 \phi(x)v(x)dx = 0\end{aligned}$$

Since ϕ was arbitrary, we conclude that $u(x) = \int_0^x v(t)dt$ a.e, so we are done.

(b). Let $\mathbf{1}_E$ denote a characteristic function on E. Then,

$$|u(x) - u(y)| = \Big| \int_0^1 u'(t)\mathbf{1}_{[x,y]} \Big|$$

By Jensen's inequality, whenever $p \geqslant 1$,

$$\int_0^1 u'(t)\mathbf{1}_{[x,y]} \leqslant \Big(\int_0^1 u'(t)^p \mathbf{1}_{[x,y]} \Big)^{\frac{1}{p}}$$

Combining all of the above:

$$
\begin{aligned}
|u(x)-u(y)| &= \Big|\int_0^1 u'(t)\mathbf{1}_{[x,y]}\Big| \leqslant \int_0^1 |u'(t)|\mathbf{1}_{[x,y]} \\
&\leqslant \Big(\int_0^1 |u'(t)|^p \mathbf{1}_{[x,y]}\Big)^{\frac{1}{p}} \\
&\leqslant \Big(\int_0^1 \mathbf{1}_{[x,y]}\Big)^{1-\frac{1}{p}} \Big(\int_0^1 |u'(t)|^p\Big)^{\frac{1}{p}} \\
&= |x-y|^{1-\frac{1}{p}} \Big(\int_0^1 |u'(t)|^p\Big)^{\frac{1}{p}}
\end{aligned} \tag{3.2}
$$

Where the final inequality is by Hölder's Inequality. Then we are done.

4. Problem 5

Choose $W \subset\subset U$ with $V \subset\subset W$. Let ϕ_ϵ denote our standard mollifier, and mollify with the characteristic function on W. Since W has compact closure, we know that $\phi_\epsilon * \mathbf{1}_W := \Phi_\epsilon(\mathbf{1}_W) \to \mathbf{1}_W$ uniformly as $\epsilon \to 0^+$. This then implies that there exists $\epsilon > 0$ such that $\Phi_\epsilon(\mathbf{1}_W)\big|_V \equiv 1$ (in this case we could choose any $0 < \epsilon < 1/2\text{dist}(W, \overline{V})$).

Then the support of this mollified function is contained in W by construction and hence this would constitute our cutoff function, since we already know the mollified function is smooth. Then we are done.

5. Problem 6

Let U have a finite cover $\{V_1, \dots, V_N\}$. Then, to each V_i, associate find a ϕ_i as constructed in the previous problem whose support is contained entirely within each V_i, $\phi_i|_U \equiv 1$, and ϕ_i is a smooth function. Then define

$$
\psi_i(x) := \frac{\phi_i(x)}{\sum_{i=1}^N \phi_i(x)}
$$

By construction, $\sum_i \psi_i \equiv 1$ at every point in U, and the support of each ψ_i is still contained in each V_i. Also, each ψ_i is clearly smooth as the ratio of smooth functions. Then, the collection $\{\psi_i\}$ satisfy the requirements and constitute a partition of unity subordinate to our given cover.

6. Problem 7

Note that $|u|^p \leqslant |u|^p \alpha \cdot \eta$. We employ the notation of an absorbing constant, where C may not be the exact same constant on each line.

$$\begin{aligned}
\int_{\partial U} |u|^p dS &\leqslant \int_{\partial U} |u|^p \alpha \cdot \eta dS \\
&= \int_U |u|^p \text{div}\alpha + p|u|^{p-1} Du \cdot \alpha dx \\
&\leqslant C(U,n) \int_U |u|^p + p|u|^{p-1}|Du| \max_i\{\alpha_i\} dx \\
&\leqslant C(U,n) \int_U p|u|^p + |Du|^p dx \quad \text{(Young's Ineq)} \\
&\leqslant C(U,n,p) \int_U |u|^p + |Du|^p dx
\end{aligned} \tag{6.1}$$

And we see that $\int_{\partial U} |u|^p dS \leqslant C \int_U |u|^p + |Du|^p dx$, as desired.

7. Problem 9

First let $u \in C_c^\infty(U)$. We have:

$$\begin{aligned}
\int_{\partial U} Du \cdot Du dx &\leqslant \int_U |u||D^2 u| dx \quad \text{(Divergence Thm)} \\
&\leqslant ||u||_{L^2(U)} ||D^2 u||_{L^2(U)} \quad \text{(Cauchy-Schwartz ineq)}
\end{aligned} \tag{7.1}$$

Taking the square root of the above,

$$||Du||_{L^2(U)} \leqslant ||u||_{L^2(U)}^{1/2} ||D^2u||_{L^2(U)}^{1/2}$$

By definition of $H_0^1(U)$, we can find a sequence $\{u_k\} \in H^1(U) \cap C_c^\infty(U)$ converging to u in $H^1(U)$. Likewise, by the smoothness of the boundary ∂U, we can extend u to a set V such that $U \subset\subset V$. Then, by density of $C_c^\infty(V)$, we can find a sequence $\{v_k\}$ in $C^\infty(\overline{U})$ such that $v_k \to u$ in $H^2(U)$. Using this,

(7.2)

$$\begin{aligned}
\Big|\int_{\partial U} (v_k - u_k)\frac{\partial v_k}{\partial \nu} dS\Big| &= \Big|\int_U D(v_k - u_k)Dv_k dx + \int_U (u_k - v_k)D^2 v_k dx\Big| \\
&\leqslant \int_U |Du_k - Dv_k||Dv_k|dx + \int_U |u_k - v_k||D^2 v_k|dx \\
&\leqslant ||Du_k - Dv_k||_{L^2(U)}||Dv_k||_{L^2(U)} + ||u_k - v_k||_{L^2(U)}||D^2 v_k||_{L^2(U)} \\
&\to 0
\end{aligned}$$

As $k \to \infty$. Note that since each u_k vanishes on the boundary, we can also compute the following:

$$\int_{\partial U} (v_k - u_k)\frac{\partial v_k}{\partial \nu} dS = \int_{\partial U} v_k \frac{\partial v_k}{\partial \nu} dS \to \int_U |Du|^2 dx + \int_U uD^2 u dx$$

By using the divergence theorem. However, on one hand, we see that this tends to 0 as well. Hence:

$$\int_U |Du|^2 dx + \int_U uD^2 u dx = 0$$

But then, identically as in the case for $u \in C_c^\infty(U)$:

$$\int_{\partial U} Du \cdot Du dx \leqslant \int_U |u||D^2 u| dx$$

(7.3)
$$\leqslant ||u||_{L^2(U)} ||D^2 u||_{L^2(U)}$$

$$\implies ||Du||_{L^2(U)} \leqslant ||u||^{1/2}_{L^2(U)} ||D^2 u||^{1/2}_{L^2(U)}$$

As asserted.

8. Problem 10

(a). Rewrite the integral as the hint says. Then, we have:

(8.1)
$$\begin{aligned}
\int_U |Du|^p dx &= \sum_i \int_U u_{x_i} u_{x_i} |Du|^{p-2} dx \\
&= \sum_i \int_U u u_{x_i x_i} |Du|^{p-2} + (p-2) u u_{x_i} |Du|^{p-4} Du \cdot Du_{x_i} dx \\
&\leqslant \sum_i \int_U |u||u_{x_i x_i}||Du|^{p-2} + (p-2)|u||u_{x_i}||Du|^{p-4}|Du||Du_{x_i}| dx \\
&\leqslant \sum_i \int_U |u||D^2 u||Du|^{p-2} + (p-2)|u||Du|^{p-2}|D^2 u| dx \\
&= n(p-1) \int_U |u||Du|^{p-2}|D^2 u| dx \\
&\leqslant n(p-1) \Big(\int_U |u|^p dx \Big)^{1/p} \Big(\int_U |Du|^p \Big)^{1-2/p} \Big(\int_U |D^2 u|^p dx \Big)^{1/p}
\end{aligned}$$

Where the above has employed Cauchy-Schwartz and Generalized Hölder's Inquality. Note that $p-1 > 0$ since $p \geqslant 2$, so our constant does not vanish. Hence by the above string of inequalities, we see that

$$\int_U |Du|^p dx \leqslant n(p-1) \Big(\int_U |u|^p dx \Big)^{1/p} \Big(\int_U |Du|^p \Big)^{1-2/p} \Big(\int_U |D^2 u|^p dx \Big)^{1/p}$$

Which implies:

$$\Big(\int_U |Du|^p dx \Big)^{2/p} \leqslant n(p-1) \Big(\int_U |u|^p dx \Big)^{1/p} \Big(\int_U |D^2 u|^p dx \Big)^{1/p}$$

And upon taking the square root:

$$||Du||_{L^p(U)} \leqslant C(n,p)||u||^{1/2}_{L^p(U)}||D^2u||^{1/2}_{L^p(U)}$$

Where $C(n,p) = (n(p-1))^{1/2}$.

(b). We proceed similarly to part a:

$$\begin{aligned}
\int_U |Du|^{2p}dx &= \sum_i \int_U u_{x_i}u_{x_i}|Du|^{2p-2}dx \\
&= \sum_i \int_U uu_{x_ix_i}\big(|Du|^2\big)^{p-1} + (2p-2)uu_{x_i}\big(|Du|^2\big)^{p-2}Du \cdot Du_{x_i}dx \\
&\leqslant n\int_U |u||D^2u|\big(|Du|^2\big)^{p-1} + (2p-2)|u|\big(|Du|^2\big)^{p-1}|D^2u|dx \\
&= n(2p-1)\int_U |u||D^2u|\big(|Du|^2\big)^{p-1}dx \\
&\leqslant n(2p-1)\int_U |u|\big(|Du|^2\big)^{p-1}dx \cdot ||D^2u||_{L^\infty(U)} \quad \text{(Hölder's)} \\
&\leqslant n(2p-1)\Big(\int_U |Du|^{2p}dx\Big)^{1-1/p}||u||_{L^p(U)}||D^2u||_{L^\infty(U)} \quad \text{(Hölder's)}
\end{aligned} \tag{8.2}$$

So that the above shows

$$\int_U |Du|^{2p}dx \leqslant n(2p-1)\Big(\int_U |Du|^{2p}dx\Big)^{1-1/p}||u||_{L^p(U)}||D^2u||_{L^\infty(U)}$$

Implying

$$\Big(\int_U |Du|^{2p}dx\Big)^{1/p} \leqslant n(2p-1)||u||_{L^p(U)}||D^2u||_{L^\infty(U)}$$

And upon taking the square root of both sides,

$$||Du||_{L^{2p}(U)} \leqslant C(n,p)||u||^{1/2}_{L^p(U)}||D^2u||^{1/2}_{L^\infty(U)}$$

As asserted (where $C(n,p) = \sqrt{n(2p-1)}$).

9. Problem 11

Consider the mollification of u, denoted u^ϵ. Since U is open and bounded, it has compact closure and hence $u^\epsilon \to u$ uniformly on U as $\epsilon \to 0$ in $L^p(U)$. Note that $Du^\epsilon = \eta_\epsilon * Du = 0$. Since $u^\epsilon \in C^\infty(U)$ and $Du^\epsilon = 0$, this implies that u is locally constant. However, U is a connected space so that any locally constant function is globally constant (since $\{x \in U \mid u = c\}$ is clopen).

Letting $\epsilon \to 0^+$, $u^\epsilon \to u$ in $L^p(U)$, so that $u^\epsilon \to u$ a.e in the standard Euclidean norm. But then, since u^ϵ is constant and tends to u a.e, we see that u is constant a.e in U.